HEATING OF A WALL OF FINITE THICKNESS BY A PERIODIC HEAT FLUX

A. M. Rushailo

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 132–134, 1966

In many practical applications (e.g., cyclical plasma accelerators, MHD generators) the cooled and uncooled walls of a channel are exposed to periodic heat fluxes, which to a first approximation can be represented by square waves [1]. The solution of a problem of this type for the case of a semi-infinite body is given in [2].

In this paper we present a solution to the problem of periodic heating of a wall of finite thickness with different cooling regimes.

In dimensionless variables, the governing equation and the initial and boundary conditions are

$$\frac{\partial\theta(\xi,\tau)}{\partial\tau}=\frac{\partial^2\theta(\xi,\tau)}{\partial\xi^2},$$

$$\left(0\leqslant\xi\leqslant 1,\quad \tau\geqslant 0,\quad \theta=\frac{T-T_0}{T_0},\quad \xi=\frac{x}{h},\quad \tau=\frac{at}{h^2}\right), \tag{1}$$

$$\theta(\xi,0)=0,\qquad \frac{\partial\theta(0,\tau)}{\partial\xi}=-\gamma(\tau),\qquad \frac{\partial\theta(1,\tau)}{\partial\xi}=-\beta\theta(1,\tau), \tag{2}$$

$$\gamma=\gamma_0\eta(\tau),\quad \gamma_0=\frac{q_0h}{T_0\lambda},\quad \beta=\frac{\alpha h}{\lambda},\quad \tau_0=\frac{at_0}{h^2},\quad \tau_1=\frac{at_1}{h^2},$$

$$\eta(\tau)=\begin{cases}1; & n\tau_1\leqslant\tau\leqslant n\tau_1+\tau_0\\ 0, & n\tau_1+\tau_0<\tau<(n+1)\tau_1,\quad n=0,1,2,\ldots\end{cases} \tag{3}$$

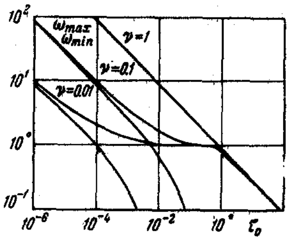

Fig. 1

Here T_0, T are the initial and the instantaneous value of the temperature, h is the wall thickness, x the coordinate, t time, t_0 the time of heat supply in one period, t_1 the period of the heat flux, q_0 = const the heat flux, λ, a, and α the thermal conductivity, thermal diffusivity, and heat transfer coefficient, τ the Fourier number, and β the Biot number.

We solve the problem by the Laplace transform method [2, 3]. The transforms $\theta^*(\xi, p)$ and $\gamma^*(p)$ are governed by the equation and boundary conditions

$$\frac{d^2\theta^*(\xi,p)}{d\xi^2}=p\theta^*(\xi,p),\qquad \frac{d\theta^*(0,p)}{d\xi}=-\gamma^*(p),$$

$$\frac{d\theta^*(1,p)}{d\xi}=-\beta\theta^*(1,p). \tag{4}$$

The solution of (4) is easily found to be

$$\theta^*(\xi,p)=\gamma^*(p)\,\theta_1^*(\xi,p),\quad \theta_1^*(\xi,p)=$$

$$=\frac{\cos[(\xi-1)i\sqrt{p}]-(\beta/i\sqrt{p})\sin[(\xi-1)i\sqrt{p}]}{\beta\cos i\sqrt{p}-i\sqrt{p}\sin i\sqrt{p}}. \tag{5}$$

The function $\theta^*_1(\xi, p)$ is a meromorph function of the complex variable p with first-order poles at the points p_k which satisfy the transcendental equation

$$z\,\mathrm{tg}\,z=\beta,\qquad p=-z^2. \tag{6}$$

Evaluating the residues of $\theta^*_1(\xi, p)$ at the points $p_k = -z_k^2$ by the second expansion theorem [3], we find the inverse transform

$$\theta_1(\xi,\tau)=2\sum_{k=1}^{\infty}F_k(\xi)\,z_k^2\exp(-z_k^2\tau),$$

$$F_k(\xi)=\frac{(z_k^2+\beta^2)\cos(\xi z_k)-2z_k\beta\sin(\xi z_k)}{z_k^2(z_k^2+\beta^2+\beta)}. \tag{7}$$

Taking account of the convolution theorem and (5), we obtain the following expression for the inverse transform of $\theta^*(\xi, p)$:

$$\theta(\xi,\tau)=\int_0^\tau\gamma(y)\,\theta_1(\xi,\tau-y)\,dy. \tag{8}$$

Substituting (7) into (8) and changing the order of integration and summation, we obtain

$$\theta(\xi,\tau)=2\gamma_0\sum_{k=1}^{\infty}F_k(\xi)\,G_k(\tau),$$

$$G_k(\tau)=z_k^2\exp(-z_k^2\tau)\int_0^\tau\eta(y)\exp(z_k^2y)\,dy. \tag{9}$$

In the following we represent the time τ in the form

$$\tau=m\tau_1+\tau_*\qquad (0\leqslant\tau_*\leqslant\tau_1,\quad m=0,1,2,\ldots). \tag{10}$$

Splitting the integral in (9) into a sum of m integrals, we find

$$G_k(\tau)=\exp(-z_k^2\tau_*)\Big\{\exp[z_k^2((\tau_*-\tau_0)\eta(\tau)+\tau_0)]-1+$$

$$+\frac{\exp(z_k^2\tau_0)-1}{\exp(z_k^2\tau_1)-1}[1-\exp(-z_k^2m\tau_1)]\Big\}. \tag{11}$$

Thus, we have obtained the solution to the problem in the form of a series (9), each term of which is a product of a function of the coordinate (7) and a function of time (11), under conditions (6) and (10), for z_k, m, and τ.

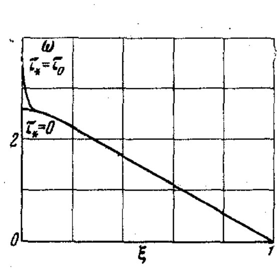

Fig. 2

We see from (10) and (11) that for all β, except $\beta = 0$ (no heat flux), at $m \gg z_1^{-2}\tau_1^{-1}$ the initial transients decay and the temperature approaches a limiting periodic cycle with the period τ_1. We shall consider several limiting cases.

1. Let $\beta = \infty$. In this case (2), (6), and (7) yield

$$\theta(1,\tau)=0,\qquad z_k={}^1/_2(2k-1)\pi,\qquad F_k(\xi)=\cos(\xi z_k)z_k^{-2}.$$

After the initial transient decays $(m \gg 2\pi^{-2}\tau_1^{-1})$ the temperature of the surface $\xi = 0$ oscillates with a period τ_1 between a maximum at $\tau_* = \tau_0$ and a minimum at $\tau_0 = 0$:

$$\theta_{max}=2\gamma_0\sum_{k=1}^{\infty}\frac{1}{z_k^2}\,\frac{1-\exp(-z_k^2\tau_0)}{1-\exp(-z_k^2\tau_1)}, \tag{12}$$

$$\theta_{\min} = 2\gamma_0 \sum_{k=1}^{\infty} \frac{1}{z_k^2} \frac{\exp(z_k^2\tau_0) - 1}{\exp(z_k^2\tau_1) - 1}. \qquad (12) \text{ (cont'd)}$$

Figure 1 shows the maximum and minimum (12) values of $\omega = \theta/\theta^0$ for $\xi = 0$ and $m = \infty$ as a function of τ_0 for $\nu = \tau_0/\tau_1$ equal to 1, 0.1, and 0.01 (the tabular data were taken from [4, 5]). The function ω is the ratio of the temperature $\theta = \theta(\xi, \tau)$ to the temperature $\theta^\circ = 2\gamma_0\tau^{1/2}\pi^{-1/2}$, which is the temperature of the surface of a semi-infinite body during the first cycle ($m = 0$, $\tau_* = \tau_0$) [1, 2].

Thus, one can see from Fig. 1 that for $m \to \infty$ even in the case of a strongly cooled wall the surface temperature can exceed the maximum temperature of the first cycle ($m = 0$) by a factor of ten or more if the value of τ_0 is low enough.

Figure 2 shows the distribution of the relative temperature $\omega(\xi) = \theta(\xi)/\theta^0$ in the wall for $\beta = m = \infty$ and two values $\tau_* = 0$ and $\tau_* = \tau_0$ and fixed values $\tau_0 = 10^{-3}$, $\tau_1 = 10^{-2}$.

2. Let $\beta = 0$. In this case (6) and (7) yield

$$z_k = (k-1)\pi, \quad F_k(\xi) = \cos(\xi z_k) z_k^{-2}.$$

Since $z_1 = 0$, we separate out the first term of the series in (9). After some elementary transformations, we can represent the temperature $\theta(\xi, \tau)$ as a sum of a function linear in time and a periodic function $\varphi(\xi, \tau)$ with period τ_1:

$$\theta(\xi, \tau) = A\tau + \varphi(\xi, \tau), \quad A = 2\gamma_0 \frac{\tau_0}{\tau_1},$$

$$\varphi(\xi, \tau) = 2\gamma_0 \left[\tau_0 \left(1 - \frac{\tau_*}{\tau_1}\right) + (\tau_* - \tau_0)\eta(\tau) + \sum_{k=2}^{\infty} \frac{\cos(\xi z_k)}{z_k^2} G_k(\tau)\right].$$

The amplitude of the function φ for $m \gg \pi^{-2}\tau_1^{-1}$ and $\xi = 0$ is

$$\varphi_0 = \varphi_{\tau_* = \tau_0} - \varphi_{\tau_* = 0} = 2\gamma_0 \left\{\tau_0 \left(1 - \frac{\tau_0}{\tau_1}\right) + \sum_{k=2}^{\infty} \frac{1}{z_k^2} \frac{1 - \exp(-z_k^2\tau_0)}{1 - \exp(-z_k^2\tau_1)} \left[1 - \frac{\exp(-z_k^2\tau_1)}{\exp(-z_k^2\tau_0)}\right]\right\}. \qquad (14)$$

The author is grateful to G. M. Bam-Zelikovich and A. B. Vatazhin for valuable discussions.

REFERENCES

1. A. M. Rushailo, "Measurement of heat fluxes and estimate of electrode temperature in a pulsed electromagnetic plasma accelerator," PMTF, no. 4, 1965.

2. G. Carslaw and D. Jaeger, Conduction of Heat in Solids [Russian translation], Izd-vo "Nauka," 1964.

3. M. A. Lavrent'ev and B. V. Shabat, Methods of the Theory of Functions of a Complex Variable [in Russian], Fizmatgiz, 1958.

4. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products [in Russian], Fizmatgiz, 1962.

5. Library of Mathematical Tables, vol. 9, Tables of Elementary Functions (sin x, cos x, e^x, e^{-x}), VTs AN SSSR, 1960.

28 June 1965 Moscow